

RIGIDITY OF PERIODIC DIFFEOMORPHISMS ON HOMOTOPY $K3$ SURFACES

JIN HONG KIM

ABSTRACT. In this paper we show that homotopy $K3$ surfaces do not admit a periodic diffeomorphism of odd prime order 3 acting trivially on cohomology. This gives a negative answer for period 3 to Problem 4.124 in the Kirby's problem list. In addition, we give an obstruction in terms of the rationality and sign of the spin numbers to the existence of a periodic diffeomorphism of odd prime order acting trivially on cohomology of homotopy $K3$ surfaces. The main strategy is to calculate the Seiberg-Witten invariant for the trivial spin^c structure in the presence of such a \mathbf{Z}_p -symmetry in two ways: (1) the new interpretation of the Seiberg-Witten invariants of M. Furuta and F. Fang, and (2) the theorem of J. Morgan and Z. Szabó on the Seiberg-Witten invariant of homotopy $K3$ surfaces for the trivial Spin^c structure. As a consequence, we derive a contradiction for any periodic diffeomorphism of prime order 3 acting trivially on cohomology of homotopy $K3$ surfaces.

1. INTRODUCTION

For compact complex surfaces X , it is a well-known question to ask whether the group $\text{Aut}(X)$ of holomorphic automorphisms of X acts faithfully on the cohomology ring $H^*(X, A)$ with values in some ring A . In general, the answer is negative if the Lie algebra of $\text{Aut}(X)$ does not reduce to $\{0\}$. For a concrete example, there exists an Enriques surface X for which $\text{Aut}(X)$ does not act faithfully on $H^2(X, \mathbf{Q})$ with values in the rational numbers. However, for a $K3$ surface X the automorphism group $\text{Aut}(X)$ is known to act faithfully on the second cohomology group $H^2(X, \mathbf{Z})$. In particular, a holomorphic cyclic \mathbf{Z}_p -action of prime order p on a $K3$ surface cannot be (co)homologically trivial. See [24] and references therein for more details.

The aim of this paper is to give a negative answer to the following Problem 4.124 in the Kirby's problem list [15] which was proposed by Allan Edmonds:

Question 1.1. *Do $K3$ surfaces admit a periodic diffeomorphism of prime order acting trivially on cohomology (or homology)?*

The answer to the Question 1.1 has already been given negatively for period 2 by T. Matumoto [19] and independently by D. Ruberman [25]. This shows the rigidity of homologically trivial actions of prime order on $K3$ surfaces.

In the present paper we attempt to answer the Question 1.1 even on *homotopy* $K3$ surfaces, not just $K3$ surfaces, by a contradiction. In a little more detail, we first assume that the spin number of a periodic diffeomorphism τ of odd prime order satisfying $b_+(X/\tau) = 3$

Date: February 2, 2008; Preliminary version.

1991 Mathematics Subject Classification. Primary 57R55.

Key words and phrases. $K3$ surfaces, periodic diffeomorphisms, Seiberg-Witten invariants.

is both rational and negative. Then we show in Theorem 4.6 that the Seiberg-Witten invariant for the trivial spin^c structure vanishes identically. To show such a vanishing result, we use the new K -theoretic interpretation of the Seiberg-Witten invariants introduced by Furuta and nicely developed by Fang. On the other hand, Morgan and Szabó have already proved in [21] that the Seiberg-Witten invariant on homotopy $K3$ surfaces for the trivial spin^c structure should be $\pm 1 \bmod 2$. This implies an obvious contradiction for odd prime period under the assumption that the spin number is both negative and rational.

In case of prime period 3, we can show further that the spin number is indeed rational (see Corollary 3.3). Moreover, it is also true that a homotopy $K3$ surface admits a periodic diffeomorphism of order 3 which acts trivially on cohomology only if its spin number is negative. Thus we can give a negative answer to Question 1.1 as follows.

Theorem 1.2. *Let X be a homotopy $K3$ surface, and let $\tau : X \rightarrow X$ be a periodic diffeomorphism of order 3. Then τ cannot act trivially on cohomology.*

Here we remark that the condition that the periodic diffeomorphism acts trivially on cohomology cannot be replaced by the condition $b_+(X/\tau) = 3$. Indeed, there exists an action of \mathbf{Z}_3 on a certain quartic $K3$ surface X in \mathbf{CP}^3 which acts trivially on $H_+^2(X; \mathbf{R})$, as S. Mukai shows on page 187 of [22]. To be precise, there exists an action of $PSL(2, \mathbf{F}_7)$ of a certain $K3$ surface in \mathbf{CP}^3 . Since the group $PSL(2, \mathbf{F}_7)$ is simple and the subgroup of $PSL(2, \mathbf{F}_7)$ which acts trivially on $H^0(X, \Omega_X^2)$ is normal, the action on $H^0(X, \Omega_X^2)$ is actually trivial. (See also [23].) Moreover, since the action of $PSL(2, \mathbf{F}_7)$ preserves the Kähler form of \mathbf{CP}^3 , it preserves that of the $K3$ surface. Thus the action on $H_+^2(X) \otimes \mathbf{C}$ is trivial. On the other hand, since $PSL(2, \mathbf{F}_7)$ contains a copy of \mathbf{Z}_3 , there exists such a \mathbf{Z}_3 action on a $K3$ surface.

It will be clear from Section 4 that we can give an obstruction to the existence of a periodic diffeomorphism of odd prime order acting trivially on cohomology of homotopy $K3$ surfaces as follows.

Theorem 1.3. *Let X be a homotopy $K3$ surface, and let $\tau : X \rightarrow X$ be a periodic diffeomorphism of odd prime order p . Assume that the spin number $\text{Spin}(\hat{\tau}, X)$ is both rational and negative. Then τ cannot act trivially on the self-dual part $H_+^2(X; \mathbf{R})$ of the second cohomology group.*

As an immediate corollary, we can give the following

Corollary 1.4. *Let X be a homotopy $K3$ surface, and let $\tau : X \rightarrow X$ be a periodic diffeomorphism of odd prime order p . Assume that the spin number $\text{Spin}(\hat{\tau}, X)$ is rational and negative. Then τ cannot act trivially on cohomology.*

For a detailed definition of the spin number $\text{Spin}(\hat{\tau}, X)$, see Section 3. In contrast to the above negative result, it is interesting to note that Edmonds [8] has constructed locally linear homologically trivial \mathbf{Z}_p -actions of odd prime order p on any closed simply connected topological 4-manifold.

Recently Chen and Kwasik announced, among many other things, a negative result in [6] to Question 1.1 under the stronger assumption that a *symplectic* action of odd prime

order on $K3$ surfaces acts trivially on cohomology. Their method is quite different from ours. In fact, they use a pseudo-holomorphic curve theory for symplectic 4-orbifolds in order to obtain an equivariant version of the Taubes' well-known theorem on symplectic 4-manifolds of [26]. Moreover, the claim that the fixed points of such symplectic actions on $K3$ surfaces like holomorphic actions are always isolated is a crucial ingredient in their proof. In general, the fixed-point set of smooth actions of odd prime order on homotopy $K3$ surfaces may or may not be isolated, 2-dimensional or a union of isolated points and 2-dimensional submanifolds. This greatly complicates the proof of Theorem 1.3 as the present paper shows. (See [6], [5] and [26] for more details.)

We organize this paper as follows. In Section 2, we set up basic notations and collect some important theorems necessary for the later sections. In Section 3, we recall the general formula for the spin numbers, and then we see that the spin number is always real. In the same section we show a very important relationship between the differences of the dimensions of the eigenspaces under the action of \mathbf{Z}_p on the representation spaces induced from the spinor vector bundles. Section 4 is devoted to the proof of Theorem 4.6 which yields immediately the proof of Theorem 1.3. In the same Section 4 we give a complete proof of Theorem 1.3. Finally in Section 5, we give a proof of Theorem 1.2.

Acknowledgements: The author is grateful to Professor Ian Hambleton for telling the work of W. Chen and S. Kwasik to him by email and Professor Weimin Chen for sending his manuscript [6].

2. PRELIMINARIES

In this section, we briefly review the Seiberg-Witten equations and the new interpretation of Seiberg-Witten invariants by Furuta and Fang. (See [11, 12, 10, 4, 14] for more details.)

2.1. Seiberg-Witten invariants: Let c be a spin^c structure on X whose associated line bundle is L , and S_{\pm} denote the positive or negative spinor vector bundles. Let \mathcal{A} be the space of connections on L , and let $\Omega^+(X)$ be the space of self-dual 2-forms. We denote by \mathcal{G} the group of gauge transformations on L . Then using the Seiberg-Witten equations we have a \mathcal{G} -equivariant map

$$f : \mathcal{A} \times \Gamma(S_+) \rightarrow i\Omega^+(X) \times \Gamma(S_-), \quad (A, \phi) \mapsto (F_A^+ - q(\phi), D_A\phi),$$

where $q(\phi) = \phi \otimes \phi^* - \frac{|\phi|^2}{2}\text{Id}$. For the sake of convenience, we assume that $\Gamma(S_+)$ and \mathcal{A} are completed with a Sobolev norm L_2^4 and that $\Gamma(S_-)$ and $\Omega^+(X)$ are completed with a Sobolev norm L_2^3 .

We fix a connection A_0 . Since $A = A_0 + a$ and the stabilizer of A_0 is S^1 , we have an S^1 -equivariant map

$$f : i\Omega_c^1(X) \times \Gamma(S_+) \rightarrow i\Omega^+(X) \times \Gamma(S_-), \quad (a, \phi) \mapsto (d^+a - q(\phi), D_{A_0}\phi + a \cdot \phi),$$

where $\Omega_c^1(X)$ is the subspace of 1-forms in the $\text{Ker } d^*$.

To construct the finite dimensional approximation of Furuta, we let $U_0 = \Gamma(S_+)$ and $U'_0 = \Gamma(S_-)$, and let $V_0 = i\Omega_c^1(X)$ and $V'_0 = i\Omega^+(X)$. For each positive real number λ , we

define U_λ (resp. U'_λ) the vector space spanned by eigenvectors of the operator $D_{A_0}^* D_{A_0}$ (resp. $D_{A_0} D_{A_0}^*$) with eigenvalues less than or equal to λ . Similarly we define V_λ and V'_λ using the elliptic operator d^+ .

Let p_λ denote the L_2 -orthogonal projection of $U_0 \oplus V_0$ onto $U'_\lambda \oplus V'_\lambda$. Using the restriction of f and the projection p_λ , we have an S^1 -equivariant map

$$f_\lambda : U_\lambda \oplus V_\lambda \rightarrow U'_\lambda \oplus V'_\lambda,$$

where S^1 acts on $\Omega^i(X)$ ($i = 1, 2$) trivially and on $\Gamma(S_\pm)$ by the complex multiplication.

Let $W'_\lambda = U'_\lambda \oplus V'_\lambda$ and $\nu_0 = -F_{A_0}^+ + \mu$ for a generic 2-form μ . Using the compactness of the Seiberg-Witten moduli space, Furuta showed the following lemma (see also Lemma 2.1-2 in [10]):

Lemma 2.1 ([12]). *For a generic parameter μ as above and a sufficiently large positive R , there exists a positive number Λ such that $f_\lambda^{-1}(\nu_0)$ and $(0 \times V_\lambda) \cap B_\lambda(R)$ do not intersect for $\lambda \geq \Lambda$, where $B_\lambda(R)$ is the ball with radius R in W_λ .*

From now on, we assume that λ is greater than or equal to Λ . Then Lemma 2.1 gives rise to an S^1 -equivariant map

$$f_\lambda : (B_\lambda, \partial B_\lambda \cup ((0 \times V_\lambda) \cap B_\lambda)) \rightarrow (W'_\lambda, W'_\lambda - B_\epsilon(\nu_0)),$$

where B 's denote balls. Now, passing to the quotients of the previous map f_λ , we get an S^1 -equivariant map

$$f_\lambda : S^{V_\lambda \oplus \mathbf{R}} \wedge S(U_\lambda) \rightarrow S^{W'_\lambda},$$

where $S^{V_\lambda \oplus \mathbf{R}}$ and $S^{W'_\lambda}$ are called the Thom spaces.

Let $m = \dim_{\mathbf{C}} U_\lambda$, $m' = \dim_{\mathbf{R}} V_\lambda$, $n = \dim_{\mathbf{C}} U'_\lambda$, and $n' = \dim_{\mathbf{R}} V'_\lambda$. Then we have

$$\frac{1}{4}(c_1(L)^2 - \sigma(X)) = 2m - 2n \quad \text{and} \quad -b_2^+(X) = m' - n'.$$

Thus we have $2d = \frac{1}{4}(c_1(L)^2 - (2\chi(X) + 3\sigma(X))) = 2m + m' - (2n + n' + 1)$. By the Thom isomorphism theorem we have the following commutative diagram

$$\begin{array}{ccc} H_{S^1}^{2n+n'}(S^{W'_\lambda}, \mathbf{Z}) & \xrightarrow{f_\lambda^*} & H_{S^1}^{2n+n'}(S^{V_\lambda \oplus \mathbf{R}} \wedge S(U_\lambda), \mathbf{Z}) \\ & & \downarrow \tau \\ & & H_{S^1}^{2(m-1-d)}(S(U_\lambda), \mathbf{Z}), \end{array}$$

where τ is the suspension isomorphism and all the cohomology appearing in the diagram means S^1 -equivariant cohomology.

Let $\Phi \in H_{S^1}^{2n+n'}(S^{W'_\lambda}, \mathbf{Z})$ be the equivariant Thom class. Then from the above diagram we have an element $\theta_{f_\lambda} \in H_{S^1}^{2(m-1-d)}(S(U_\lambda), \mathbf{Z})$ such that

$$f_\lambda^*(\Phi) = \tau^{-1}(\theta_{f_\lambda}).$$

Since S^1 acts freely on $S(U_\lambda)$, we see that

$$H_{S^1}^{2(m-1-d)}(S(U_\lambda), \mathbf{Z}) = H^{2(m-1-d)}(\mathbf{CP}^{m-1}, \mathbf{Z}).$$

Note also that since $H^*(\mathbf{CP}^{m-1}, \mathbf{Z}) = \mathbf{Z}[x]/x^m$, we can write $\theta_{f_\lambda} = a_{X,c} x^{m-1-d}$ for some $a_{X,c} \in \mathbf{Z}$.

A cyclic \mathbf{Z}_q -action on X (q is not necessarily prime) is called a *spin^c action* (or *preserves a spin^c structure*) if the generator of the action $g : X \rightarrow X$ lifts to a spin^c bundle $\hat{g} : P_{\text{Spin}^c} \rightarrow P_{\text{Spin}^c}$. Such an action is of even type if \hat{g} has order q and is of odd type if \hat{g} has order $2q$. In particular, if the spin^c bundle is a spin bundle, the action is called a *spin action*. According to [10], a cyclic \mathbf{Z}_p -action of odd prime p is of even type if and only if the associated line bundle to the spin^c structure is a \mathbf{Z}_p - $U(1)$ -bundle. Since the line bundle associated to the trivial spin^c structure is always a \mathbf{Z}_p - $U(1)$ -bundle, any cyclic spin action of odd prime order is of even type.

From now on, we denote by $\hat{\mathbf{Z}}_q$ the group generated by \hat{g} . Thus if X has an action \mathbf{Z}_q preserving the spin^c structure c , then we have an $S^1 \times \hat{\mathbf{Z}}_q$ -equivariant map $f = f_\lambda : S^{V_\lambda \oplus \mathbf{R}} \wedge S(U_\lambda) \rightarrow S^{W'_\lambda}$. Moreover, by suspending f by $S^{V_\lambda \oplus \mathbf{R}}$ we obtain a map, still denoted f ,

$$(2.1) \quad f : S^{(V_\lambda \oplus \mathbf{R}) \otimes \mathbf{C}} \wedge S(U_\lambda) \rightarrow S^{W'_\lambda \oplus V_\lambda \oplus \mathbf{R}}.$$

Applying the $K_{S^1 \times \hat{\mathbf{Z}}_q}$ -theory we get a $\beta_f \in K_{S \times \hat{\mathbf{Z}}_q}(S(U_\lambda))$ such that

$$f^*(\tau_{W'_\lambda \oplus V_\lambda \oplus \mathbf{R}}) = \beta_f \tau_{(V_\lambda \oplus \mathbf{R}) \otimes \mathbf{C}},$$

where $\tau_{W'_\lambda \oplus V_\lambda \oplus \mathbf{R}}$ and $\tau_{(V_\lambda \oplus \mathbf{R}) \otimes \mathbf{C}}$ are the K -theory Thom classes. Then we can summarize the results of Furuta and Fang in the following theorem (see Theorems 2.3 and 2.4 and Proposition 4.3 in [10]).

Theorem 2.2 (Furuta, Fang). *Let X be a smooth 4-manifold with $b_1(X) = 0$ and that $b_2^+(X) \geq 2$. Let c denote spin^c structure on X .*

- (1) *For a sufficiently large $\lambda \geq \Lambda$, the Seiberg-Witten invariants satisfy $SW(X, c) = a_{X,c}$. Furthermore, if X has an action \mathbf{Z}_q preserving the spin^c structure c and $H_+^2(X/\mathbf{Z}_q, \mathbf{R}) \neq 0$, then there exists an $S^1 \times \hat{\mathbf{Z}}_q$ -equivariant map*

$$f_\lambda : S^{V_\lambda \oplus \mathbf{R}} \wedge S(U_\lambda) \rightarrow S^{W'_\lambda}$$

and $SW(X, c) = a_{X,c}$.

- (2) *Let t be the standard 1-dimensional complex representation of S^1 , and let $T = 1 - t$. Then $\beta = \beta_f$ satisfies the following identity:*

$$\beta_f(t) = (-1)^n a_{X,c} \left(\frac{\log(1+T)}{T} \right)^{\frac{1}{2}(b_+(M)-1)} T^{m-d-1}, \quad T^{m-d} = 0.$$

In particular, we have $\beta = \pm SW(X, c) T^{m-d-1}$.

2.2. The tom Dieck's character formula: We also need to use the tom Dieck's formula in Section 3. we explain it briefly. Since any spin cyclic actions of odd prime order are of even type, we consider spin actions only of even type.

Recall first that the group $\text{Pin}(2)$ has one non-trivial one dimensional representation $\tilde{1}$ and has a countable series of 2-dimensional irreducible representations h_1, h_2, \dots . In particular, the representation $h_1 = h$ is the restriction of the standard representation of $SU(2)$ to $\text{Pin}(2) \subset SU(2)$. The representation ring $R(\mathbf{Z}_p)$ is isomorphic to the group ring

$\mathbf{Z}(\mathbf{Z}_p)$ which is generated by the standard one-dimensional representation ξ . Thus as a \mathbf{Z} -module, $R(\mathbf{Z}_p)$ is generated by $1, \xi, \dots, \xi^{p-1}$.

For the sake of simplicity, let $V = (U_\Lambda \oplus V_\Lambda) \otimes_{\mathbf{R}} \mathbf{C}$, $W = (U'_\Lambda \oplus V'_\Lambda) \otimes_{\mathbf{R}} \mathbf{C}$, and $G = \text{Pin}(2) \times \mathbf{Z}_p$. In the presence of a spin action \mathbf{Z}_p of odd prime order p , the complex index of D is given by

$$[V] - [W] = k(\xi)h - t(\xi)\tilde{1} \in R(G),$$

where $k(\xi) = k_0 + k_1\xi + \dots + k_{p-1}\xi^{p-1}$, $t(\xi) = t_0 + t_1\xi + \dots + t_{p-1}\xi^{p-1}$ with the properties $t_0 + t_1 + \dots + t_{p-1} = b_+(X)$ and $k_0 + k_1 + \dots + k_{p-1} = -\frac{\sigma(X)}{8}$.

Let BV and BW denote balls in V and W . Then it follows from Lemma 2.1 that there exists a G -equivariant map f preserving the boundaries SV and SW of BV and BW , respectively:

$$f = f_\Lambda : (BV, SV) \rightarrow (BW, SW).$$

Let $K_G(V)$ denote $K_G(BV, SV)$. Similarly define $K_G(W)$. Then $K_G(V)$ (resp. $K_G(W)$) is a free $R(G)$ -modules with one generator $\lambda(V)$ (resp. $\lambda(W)$), called the *Bott class*. Now, applying K -theory functor to f we get a map

$$f^* : K_G(W) \rightarrow K_G(V)$$

with a unique element α_f , called the *K-theoretic degree of f* , satisfying the equation

$$(2.2) \quad f^*(\lambda(W)) = \alpha_f \cdot \lambda(V).$$

Let V_g and W_g denote the subspaces of V and W fixed by an element $g \in G$, and V_g^\perp and W_g^\perp denote the their orthogonal complements. Let $f^g : V_g \rightarrow W_g$ be the restriction of f to V_g , and let $d(f^g)$ denote the topological degree of f^g . Then the tom Dieck's character formula says that we have

$$\text{tr}_g(\alpha_f) = d(f^g) \text{tr}_g \left(\sum_{i=0}^{\infty} (-1)^i \Lambda^i(W_g^\perp - V_g^\perp) \right),$$

where tr_g is the trace of the action of an element $g \in G$. Note also that the topological degree $d(f^g)$ is by definition zero, if $\dim(V_g) \neq \dim(W_g)$.

3. CYCLIC GROUP ACTIONS AND SPIN NUMBERS

In this section, we prove the important lemmas about the spin numbers which are essential to prove our main result.

As before, let X denote a homotopy $K3$ surface and let τ be a periodic diffeomorphism of odd prime order p , unless stated otherwise. Let $\sigma(X)$ denote the signature of X . As remarked in Subsection 2.1, it suffices to consider spin actions τ of even type.

First recall that the spin number for the lifting $\hat{\tau}$ is defined to be

$$\text{Spin}(\hat{\tau}, X) = \text{ind}_{\hat{\tau}} D = \text{tr}(\hat{\tau}|_{\ker D}) - \text{tr}(\hat{\tau}|_{\text{coker } D}),$$

where D denotes the Dirac operator as before. These spin numbers can be calculated in terms of the fixed point set X^τ by the general Lefschetz formula (Theorem 3.9 in [3]).

In more detail, for each $x \in X^\tau$, the fiber N_x^τ of the normal bundle N^τ is a real \mathbf{Z}_p -module. Since \mathbf{Z}_p is cyclic of odd prime order, its real irreducible representation is of the form

$$\tau \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

It is important to note that the above representation given by θ and $-\theta$ are equivalent. Thus the fiber N_x^τ of the normal bundle has a canonical decomposition

$$N_x^\tau = \sum_{\mu} N_x^\tau(\mu),$$

where μ 's are the complex numbers of absolute value 1 with positive imaginary part and where $N_x^\tau(\mu)$ has a complex structure in which τ acts by multiplication with μ . Then we have the following Lefschetz theorem about the spin numbers.

Theorem 3.1. *Let $\tau : X \rightarrow X$ be an isometry of the homotopy K3 surface X of odd prime order p . Let the fixed-point set X^τ consist of isolated points $\{P_j\}$ and connected 2-manifolds $\{F_k\}$. For each j , let the action of τ on the tangent space at P_j be given by the matrix*

$$\begin{pmatrix} \cos \alpha_j & -\sin \alpha_j \\ \sin \alpha_j & \cos \alpha_j \end{pmatrix} \oplus \begin{pmatrix} \cos \beta_j & -\sin \beta_j \\ \sin \beta_j & \cos \beta_j \end{pmatrix}$$

relative to an oriented basis, where α_j and β_j denote $2\pi l_{\alpha_j}/p$ and $2\pi l_{\beta_j}/p$ ($0 < \alpha_j, \beta_j < \pi$), respectively. For each k , let τ act on the normal bundle N_k^τ of Y_k by multiplication with $e^{i\theta_k}$, where $\theta_k = 2\pi l_{\theta_k}/p$ ($0 < \theta_k < \pi$). Let $\hat{\tau}$ be a lifting of τ preserving the trivial spin^c structure. Then we have the following formula for spin number:

$$(3.1) \quad \begin{aligned} \text{Spin}(\hat{\tau}, X) = & -\frac{1}{4} \sum_{P_j} \epsilon(P_j, \hat{\tau}) \csc(\alpha_j/2) \csc(\beta_j/2) \\ & + \frac{1}{4} \sum_{F_k} \epsilon(F_k, \hat{\tau}) \cos(\theta_k/2) \csc^2(\theta_k/2) \langle [F_k], [F_k] \rangle, \end{aligned}$$

where $\epsilon(P_j, \hat{\tau})$ and $\epsilon(F_k, \hat{\tau})$ are ± 1 , depending on the action of τ on the spin bundle.

The proof of the above theorem follows from the Atiyah-Singer G -spin theorem in the literature (e.g., see [13], [2], Theorem 8.35 in [1] and Theorem 14.11 in [18]).

Then as a corollary we can show the following result.

Corollary 3.2. *Let X be a homotopy K3 surface, and let τ be a periodic diffeomorphism of odd prime order p on X . Let $\hat{\tau}$ be a lifting of τ preserving the trivial spin^c structure. Then the following holds:*

- (1) *The spin number $\text{Spin}(\hat{\tau}, X)$ is always real.*
- (2) *For each $j = 1, 2, \dots, p-1$, two spin numbers $\text{Spin}(\hat{\tau}^j, X)$ and $\text{Spin}(\hat{\tau}^{p-j}, X)$ equal to each other.*

Proof: The proof (a) is immediate from Theorem 3.1.

For the proof of (b), as remarked earlier it suffices to note that the representation of the normal bundle of a fixed-point set given by θ and $-\theta$ are equivalent. \square

As another immediate corollary, we have the following result. This will be useful to prove Theorem 1.2 which is the case for prime order 3.

Corollary 3.3. *Let X be a homotopy $K3$ surface, and let τ be a periodic diffeomorphism of odd prime order 3 on X . Let $\hat{\tau}$ be a lifting of τ preserving the trivial spin^c structure. Then the spin number $\text{Spin}(\hat{\tau}, X)$ is always rational.*

Proof: If the prime order p equals 3 then all of α_j , β_j , and θ_k are just $2\pi/3$. But then $\csc(\alpha_j/2) \csc(\beta_j/2)$ is just $\frac{4}{3}$ and $\cos(\theta_k/2) \csc^2(\theta_k/2)$ is $\frac{2}{3}$. Hence it follows from (3.1) that the spin number $\text{Spin}(\hat{\tau}, X)$ should be

$$\text{Spin}(\hat{\tau}, X) = \frac{1}{6} \sum_{F_k} \pm \langle [F_k], [F_k] \rangle + \frac{1}{3} \sum_{P_j} \pm 1,$$

which is rational, as required. \square

Next, let m_i ($i = 0, 1, 2, \dots, p-1$) denote the dimensions of the ν^i -eigenspaces of a generator of the \mathbf{Z}_p -action on U_Λ , where ν is a generator of \mathbf{Z}_p . We also define n_i in a similar way for U'_Λ . Thus $k_i = m_i - n_i$ for $i = 0, 1, 2, \dots, p-1$. Then we need the following series of lemmas:

Lemma 3.4. *Let X be a homotopy $K3$ surface, and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_+(X/\tau) = 3$. Then we have $k_0 \leq 2$.*

Proof: Since the difference of two spin structures is naturally an element of $H^1(X, \mathbf{Z}_2) = 0$, τ clearly generates a spin action of odd prime order p that is of even type.

From the previous section, recall

$$[V] - [W] = k(\xi)h - t(\xi)\tilde{1},$$

where $k(\xi) = k_0 + k_1\xi + \dots + k_{p-1}\xi^{p-1}$, $t(\xi) = t_0 + t_1\xi + \dots + t_{p-1}\xi^{p-1}$ with the properties

$$t_0 + t_1 + \dots + t_{p-1} = b_+(X) = 3,$$

$$k_0 + k_1 + \dots + k_{p-1} = -\frac{\sigma(X)}{8} = 2.$$

Since $b_+(X) = b_+(X/\tau)$ and τ is of odd prime order, we should have $t_0 = b_+(X/\tau) = 3$, and so $t_1 + \dots + t_{p-1} = 0$. Let $\alpha = \alpha_f$ be the K -theoretic degree of f in (2.2) of the form

$$\alpha_f = \alpha_0(\xi) + \tilde{\alpha}_0(\xi)\tilde{1} + \sum_{i=1}^{\infty} \alpha_i(\xi)h_i$$

Now, we want to compute $\alpha = \alpha_f$. To do so, note first that θ and $\theta\nu$ act non-trivially on h and trivially on $\tilde{1}$, where $\theta \in S^1$ is an element generating a dense subgroup of S^1 and $\nu \in \mathbf{Z}_p$ is a generator as before. Thus we have

$$\dim V_\theta - \dim W_\theta = -(t_0 + t_1 + \dots + t_{p-1}) = -b_+(X) < 0$$

$$\dim V_{\theta\nu} - \dim W_{\theta\nu} = -t_0 = -b_+(X/\tau) = -b_+(X) < 0.$$

Hence the topological degrees $d(f^\theta) = d(f^{\theta\nu}) = 0$, so we obtain $\text{tr}_\theta(\alpha) = \text{tr}_{\theta\nu}(\alpha) = 0$. This implies

$$\alpha_0(\nu) + \tilde{\alpha}_0(\nu) = 0, \quad \alpha_i(\nu) = 0 \quad i \geq 1.$$

Therefore $\tilde{\alpha}_0 = -\alpha_0$ and $\alpha_i = 0$ for all $i \geq 1$, i.e., $\alpha = \alpha_0(\xi)(1 - \tilde{1})$.

Set $\alpha_0(\xi) = a_0 + a_1\xi + \dots + a_{p-1}\xi^{p-1}$. Since $J\nu$ acts non-trivially on $\xi\tilde{1}, \xi h$, and h , we have $\dim V_{J\nu} - \dim W_{J\nu} = 0$. Thus, by definition, $d(f^{J\nu}) = 1$. Note also that

$$(3.2) \quad \text{tr}_{J\nu}(\alpha) = \text{tr}_{J\nu}(\alpha_0(\xi)(1 - \tilde{1})) = 2(a_0 + a_1\nu + \dots + a_{p-1}\nu^{p-1}).$$

On the other hand, for each $j = 1, 2, \dots, p-1$ we get

$$(3.3) \quad \begin{aligned} & \text{tr}_{J\nu^j} \left(\sum_{i=0}^{\infty} (-1)^i \Lambda^i((t_0 + t_1\xi + \dots + t_{p-1}\xi^{p-1})\tilde{1} \right. \\ & \quad \left. - (k_0 + k_1\xi + \dots + k_{p-1}\xi^{p-1})h) \right) \\ &= \text{tr}_{J\nu^j} \left((1 - \tilde{1})^{t_0} (1 - \xi^j\tilde{1})^{t_1} \dots (1 - \xi^{j(p-1)}\tilde{1})^{t_{p-1}} \right. \\ & \quad \left. \cdot (2 - h)^{-k_0} (1 + \xi^{2j} - \xi h)^{-k_1} \dots (1 + \xi^{j(p-2)} - \xi^{j(p-1)}h)^{-k_{p-1}} \right) \\ &= 2^{3-k_0} (1 + \nu^j)^{t_1} (1 + \nu^{2j})^{t_2} \dots (1 + \nu^{j(p-1)})^{t_{p-1}} \\ & \quad \cdot (1 + \nu^{2j})^{-k_1} (1 + \nu^{3j})^{-k_2} \dots (1 + \nu^{p-2j})^{-k_{p-1}}. \end{aligned}$$

where we used $\text{tr}_J(\tilde{1}) = -1$, and $\text{tr}_J(h) = 0$ in the second equality. Hence it follows from (3.2) and (3.3) that $k_0 \leq 2$. Indeed, if we multiply $p-1$ equations from (3.1), we obtain

$$\begin{aligned} & 2^{(p-1)(3-k_0)} \left(\prod_{j=1}^{p-1} (1 + \nu^j) \right)^{t_1} \dots \left(\prod_{j=1}^{p-1} (1 + \nu^{p-2j}) \right)^{-k_{p-1}} \\ &= 2^{p-1} \prod_{j=1}^{p-1} \left(\sum_{i=1}^{p-1} a_i \nu^{ji} \right). \end{aligned}$$

Since $\prod_{j=1}^{p-1} (1 + \nu^j) = 1$, the above equation should be of the form

$$\begin{aligned} 2^{(p-1)(2-k_0)} &= \prod_{j=1}^{p-1} \left(\sum_{i=1}^{p-1} a_i \nu^{ji} \right) \\ &= c_0 + c_1\nu + \dots + c_{p-1}\nu^{p-1}, \end{aligned}$$

where c_0, c_1, \dots, c_{p-1} are some integers. Thus, if $k_0 \geq 3$ then we have

$$1 = 2^{(p-1)(k_0-2)} (c_0 + c_1\nu + \dots + c_{p-1}\nu^{p-1}) \equiv 0, \quad \text{mod } 2,$$

which is a contradiction. This completes the proof. \square

Remark 3.5. Let X be a homotopy $K3$ surface, and X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_+(X/\tau) = 3$ as in Lemma 3.4. Then we can also use the lifting $e^{\frac{2\pi qi}{p}} \hat{\tau}$ instead of $\hat{\tau}$, where $0 \leq q < p$. Let m_i^q ($i = 0, 1, 2, \dots, p-1$) denote the dimensions of the ν^{i+q} -eigenspaces of a generator of the \mathbf{Z}_p -action on U_Λ , where ν is a generator of \mathbf{Z}_p . We also define n_i^q in a similar way for U'_Λ . Thus we have $k_i^q = m_i^q - n_i^q$ for $i = 0, 1, 2, \dots, p-1$. Then applying the same arguments as in Lemma 3.4 implies $k_0^q \leq 2$ for each $0 \leq q < p$. Since $k_i^q = k_{i+q}$ (here the subscripts are labelled modulus p), we can conclude that $k_i \leq 2$ for all $0 \leq i \leq p-1$.

If the spin number is rational and non-negative, we can describe k_i more precisely as follows.

Lemma 3.6. *Let X be a homotopy K3 surface, and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_+(X/\tau) = 3$. Assume that the spin number $\text{Spin}(\hat{\tau}, X)$ is both rational and non-negative. Then we have*

$$k_0 = 2 \text{ and } k_1 = k_2 = \dots = k_{p-1} = 0.$$

Proof: We continue to use the same notations as in the proof of Lemma 3.4.

To show it, note first that we have

$$(3.4) \quad \text{Spin}(\hat{\tau}, X) = k_0 + k_1\nu + \dots + k_{p-1}\nu^{p-1}.$$

Now, if we use the formula (3.4) for the spin number $\text{Spin}(\hat{\tau}, X)$ and the relation $1 + \nu + \nu^2 + \dots + \nu^{p-1} = 0$, we obtain

$$(3.5) \quad \begin{aligned} \text{Spin}(\hat{\tau}, X) &= k_0 + k_1\nu + \dots + k_{p-1}\nu^{p-1} \\ &= k_0 + k_1\nu + \dots + k_{p-2}\nu^{p-2} - k_{p-1}(1 + \nu + \dots + \nu^{p-2}) \\ &= k_0 - k_{p-1} + (k_1 - k_{p-1})\nu + \dots + (k_{p-2} - k_{p-1})\nu^{p-2}. \end{aligned}$$

Since the spin number $\text{Spin}(\hat{\tau}, X)$ is rational by assumption, the equation (3.5) has rational coefficients. Thus all the coefficients of (3.5) should vanish, since the polynomial $x^{p-1} + x^{p-2} + \dots + x + 1$ is irreducible over rational numbers. In particular, we have

$$k_1 = k_2 = \dots = k_{p-1},$$

as required.

Note also from the equation (3.4) and the relation $1 + \nu + \dots + \nu^{p-1} = 0$ that we have

$$\begin{aligned} (p-1)k_0 &= (p-1)k_1 + (p-1) \cdot \text{Spin}(\hat{\tau}, X) \\ &= 2 - k_0 + (p-1) \cdot \text{Spin}(\hat{\tau}, X). \end{aligned}$$

Thus the spin number satisfies

$$(3.6) \quad 0 \leq \text{Spin}(\hat{\tau}, X) = \frac{pk_0}{(p-1)} - \frac{2}{(p-1)}$$

and so we have $\frac{2}{p} \leq k_0 \leq 2$. But if $k_0 = 1$ then $(p-1)k_1 = 1$, which is clearly a contradiction. Hence we should have $k_0 = 2$, which implies that $k_1 = k_2 = \dots = k_{p-1} = 0$. This completes the proof of Lemma 3.6. \square

Now it is immediate to obtain the following corollary.

Corollary 3.7. *Let X be a homotopy K3 surface, and let X admit a periodic diffeomorphism τ of order 3 satisfying $b_+(X/\tau) = 3$. Assume that the spin number $\text{Spin}(\hat{\tau}, X)$ is non-negative. Then we have*

$$k_0 = 2 \text{ and } k_1 = k_2 = 0.$$

Proof: If the prime order p equals 3 then the spin number $\text{Spin}(\hat{\tau}, X)$ is rational by Corollary 3.3. Thus the equation (3.5) has rational coefficients. Thus all the coefficients of (3.5) should vanish. Therefore, we have $k_1 = k_2$, as required. \square

Next we deal with the case that the spin number is negative.

Lemma 3.8. *Let X be a homotopy $K3$ surface, and let X admit a periodic diffeomorphism τ of odd prime order p satisfying $b_+(X/\tau) = 3$. Assume that the spin number $\text{Spin}(\hat{\tau}, X)$ is both rational and negative. Then we have*

$$k_0 \leq 0 \text{ and } k_1 = k_2 = \dots = k_{p-1} = \frac{2 - k_0}{p - 1} \geq 1.$$

Moreover, the sum $k_0 + k_1 + \dots + k_{p-2}$ is equal to $2 - k_1$ which is non-negative.

Proof: From the relation (3.6), note that

$$0 > \text{Spin}(\hat{\tau}, X) = \frac{pk_0}{(p-1)} - \frac{2}{(p-1)}.$$

Thus we have $k_0 < 2/p$, i.e., $k_0 \leq 0$. The other relation concerning k_1, k_2, \dots, k_{p-1} are immediate from the identity $k_1 = k_2 = \dots = k_{p-1}$, $k_0 + k_1 + \dots + k_{p-1} = 2$, and Remark 3.5. \square

In particular, if $p = 3$ then $k_0 = 2 - 2k_1 \geq -2$. Thus under the assumption that the spin number is rational and negative we have either $k_0 = 0, k_1 = k_2 = 1$ or $k_0 = -2, k_1 = k_2 = 2$. This will be used in the proof of Proposition 4.1.

4. VANISHING THEOREMS

The aim of this section is prove the main Theorem 1.3. We do so by proving a vanishing theorem analogous to a theorem of F. Fang in [10].

Before proving the Proposition 4.1, we first need to consider the case that the spin number is both rational and non-negative. In this case, it follows from Lemma 3.6 that $k_0 = 2$ and $k_1 = k_2 = \dots = k_{p-1} = 0$. But these data are exactly what we can obtain for the trivial action on a homotopy $K3$ surface. Hence we cannot apply the arguments of this section to obtain a new calculation of the Seiberg-Witten invariants, but only the relation $\beta_f = a(1 - t)^{m-1}$ for some integer a holds. In fact, if the arguments work in this case, we would have the conclusion that there is no action of a cyclic group of odd prime order 3 on a homotopy $K3$ surface that acts trivially on H_+^2 . But there do exist examples of holomorphic actions of \mathbf{Z}_3 on a projective $K3$ surface which is trivial on H_+^2 , as we have already seen in Section 1. On the other hand, the assumption that the spin number is negative implies that the action is indeed non-trivial, and the method of this section does not make any contradiction to the known calculation of Morgan and Szabó for homotopy $K3$ surfaces in [21].

Now we prove the following proposition which will be crucial throughout this paper.

Proposition 4.1. *Let M be a smooth closed oriented spin 4-manifold with $b_1(M) = 0$ and $b_+(M) \geq 2$. Let τ generate a spin action of odd prime order p such that $b_+(M) = b_+(M/\tau)$. Assume that the virtual dimension of the Seiberg-Witten moduli space is zero. If $k_0 \leq l = \frac{1}{2}(b_+(M) - 1)$ and $k_1 = k_2 = \dots = k_{p-1}$ then we have*

$$\beta_f = a(1 + \xi + \dots + \xi^{p-1})(1 - t)^{m_0-1}(1 - t\xi)^{m_1} \dots (1 - t\xi^{p-1})^{m_{p-1}}, \quad a \in \mathbf{Z}.$$

- Remark 4.2.* (1) By the relation $k_0 + k_1 \dots + k_{p-1} = 2$, the assumption of the Proposition 4.1 implies $k_1 = k_2 = \dots = k_{p-1} = \frac{2-k_0}{p-1}$.
- (2) We assume in the statement of Proposition 4.1 that the virtual dimension d of the Seiberg-Witten moduli space is zero, since this is the only case we have for the trivial spin^c structure on homotopy $K3$ surfaces. For the method how to deal with the general case, you can see the proof of Theorem 1 in Section 4 of [10]. Indeed, for the case that the virtual dimension d of the Seiberg-Witten moduli space is not necessarily zero, we need to replace m by $m - d$ in the proof below and to consider the image of β_f in the truncated ring

$$\frac{R(S^1 \times \mathbf{Z}_p)}{(1-t)^{m_0-d}(1-t\xi)^{m_1} \dots (1-t\xi^{p-1})^{m_{p-1}}}.$$

Then we see that the proof below can be repeated *verbatim* without any difficulty.

Proof: For the sake of simplicity, we give a proof only for the case $p = 3$. This will not only greatly simplify the notational complications in the proof, but also convey our idea more quickly. The other case is completely similar. Refer to Section 5 of [10] to see how to deal with the general case in more detail.

Recall that our spin action is of even type. Let us denote by $f = f_\Lambda$ the $S^1 \times \mathbf{Z}_3$ -equivariant map

$$f : S^{(V_\Lambda \oplus \mathbf{R}) \otimes \mathbf{C}} \wedge S(U_\Lambda) \rightarrow S^{W'_\Lambda \oplus V_\Lambda \oplus \mathbf{R}}.$$

which is induced from the Seiberg-Witten equations as in (2.1).

Now, applying the Adams ψ -operation to the K -theoretic degree $\beta \in K_{S^1 \times \mathbf{Z}_3}(S(U_\Lambda))$ of f , we get

$$(4.1) \quad \begin{aligned} \psi^q(\beta) = & q^l \beta \cdot (1+t+\dots+t^{q-1})^{n_0} \cdot (1+t\xi+\dots+t^{q-1}\xi^{q-1})^{n_1} \\ & \cdot (1+t\xi^2+\dots+t^{q-1}\xi^{2(q-1)})^{n_2}. \end{aligned}$$

We will need to use the following lemma in [10]:

Lemma 4.3.

$$K_{S^1 \times \mathbf{Z}_3}(S(U_\Lambda)) = \frac{R(S^1 \times \mathbf{Z}_3)}{(1-t)^{m_0}(1-t\xi)^{m_1}(1-t\xi^2)^{m_2}},$$

where $R(S^1 \times \mathbf{Z}_3)$ denotes the representation ring of $S^1 \times \mathbf{Z}_3$.

Then we can show the following lemma:

Lemma 4.4. *There exists $\beta^{(1)} \in R(S^1 \times \mathbf{Z}_3)/(1-t\xi)^{m_1}(1-t\xi^2)^{m_2}$ such that $\beta = \beta^{(1)}(1-t)^{m_0}$.*

Proof: Let $\beta = \sum_i \left(\sum_{j=0}^2 a_j^i \xi^{2j} \right) T^i$ for $T = 1 - t$. Applying the identity (4.1) with $q = 2$, we get

$$\begin{aligned} \psi^2(\beta) &= \sum_i \left(\sum_{j=0}^2 a_j^i \xi^{2j} \right) (2T - T^2)^i \\ &= 2^l \sum_i \left(\sum_{j=0}^2 a_j^i \xi^j \right) T^i (2 - T)^{n_0} (1 + \xi - T\xi)^{n_1} (1 + \xi^2 - T\xi^2)^{n_2}. \end{aligned}$$

If we compare the coefficients of T^i , we get

$$(4.2) \quad 2^i \left(\sum_{j=0}^2 a_j^i \xi^{2j} \right) = 2^{l+n_0} (1 + \xi)^{n_1} (1 + \xi^2)^{n_2} \left(\sum_{j=0}^2 a_j^i \xi^j \right).$$

Now, put $\xi = 1$ in the equation (4.2). Then we get

$$2^i \left(\sum_{j=0}^2 a_j^i \right) = 2^{l+n} \left(\sum_{j=0}^2 a_j^i \right),$$

where $n = n_0 + n_1 + n_2$. Thus if $i < l + n$, $\sum_{j=0}^2 a_j^i = 0$. Since the virtual dimension d of the Seiberg-Witten moduli space is zero and so $l + n = m - 1 = (m_0 - 1) + m_1 + m_2 \geq m_0$, we conclude that $\sum_{j=0}^2 a_j^i = 0$ for all $i \leq m_0 - 1$. In fact, we can show that $a_0^i = a_1^i = a_2^i = 0$ for all $i \leq m_0 - 1$. To see it, we use an argument of Fang in [10]. Using the relation $(1 + \nu)(1 + \nu^2) = 1$ and the equation (4.2), we obtain

$$2^{2i} \prod_{k=1}^2 \left(\sum_{j=0}^2 a_j^i \nu^{2kj} \right) = 2^{2(l+n_0)} \prod_{k=1}^2 \left(\sum_{j=0}^2 a_j^i \nu^{kj} \right).$$

Since $\prod_{k=1}^2 \left(\sum_{j=0}^2 a_j^i \nu^{2kj} \right) = \prod_{k=1}^2 \left(\sum_{j=0}^2 a_j^i \nu^{kj} \right)$, there exist a k such that $\sum_{j=0}^2 a_j^i \nu^{kj} = 0$, provided $i < l + n_0$. Thus we should have $a_0^i = a_1^i = a_2^i = 0$ for all $i \leq m_0 - 1$. Hence the image of β in $R(S^1 \times \mathbf{Z}_3)/(1 - t)^{m_0}$ is zero. Thus it follows from Lemma 4.3 and a simple argument similar to Lemma 4.2 in [10] that there exist a $\beta^{(1)} \in R(S^1 \times \mathbf{Z}_3)/(1 - t\xi)^{m_1}(1 - t\xi^2)^{m_2}$ such that $\beta = \beta^{(1)}(1 - t)^{m_0}$. This completes the proof. \square

Similarly, we can show the following lemma whose proof is quite similar to the case above.

Lemma 4.5. *There exists $\beta^{(2)} \in R(S^1 \times \mathbf{Z}_3)/(1 - t\xi^2)^{m_2}$ such that $\beta = \beta^{(2)}(1 - t)^{m_0}(1 - t\xi)^{m_1}$.*

Proof: Let $\beta^{(1)} = \sum_i \left(\sum_{j=0}^2 b_j^i \xi^j \right) S^i$ for $S = 1 - t\xi$. Applying the identity (4.1) with $q = 2$ again, it is easy to get

$$\begin{aligned} &(1 + \xi^2 - S\xi^2)^{k_0-1} \left(\sum_i \left(\sum_{j=0}^2 b_j^i \xi^{2j} \right) (2S - S^2)^i \right) \\ &= 2^l \left(\sum_i \left(\sum_{j=0}^2 b_j^i \xi^j \right) S^i \right) (2 - S)^{n_1} (1 + \xi - S\xi)^{n_2}. \end{aligned}$$

Now, comparing the coefficients of S^i , we get

$$(4.3) \quad 2^i(1 + \xi^2)^{k_0-1} \left(\sum_{j=0}^2 b_j^i \xi^{2j} \right) = 2^{l+n_1}(1 + \xi)^{n_2} \left(\sum_{j=0}^2 b_j^i \xi^j \right).$$

By putting $\xi = 1$ in the equation (4.3), we get

$$2^{k_0+i-1} \left(\sum_{j=0}^2 b_j^i \right) = 2^{l+n_1+n_2} \left(\sum_{j=0}^2 b_j^i \right).$$

Thus if $k_0 + i - 1 < l + n - n_0$, $\sum_{j=0}^2 b_j^i = 0$. Since $l + n = m - 1 = (m_0 - 1) + m_1 + m_2$, the inequality $k_0 + i - 1 < l + n - n_0$ is equivalent to $i < m_1 + m_2$. Thus we can conclude that $\sum_{j=0}^2 b_j^i = 0$ for all $i \leq m_1 - 1$. In fact, an argument similar to the previous case shows that $b_0^i = b_1^i = b_2^i$ for all $i \leq m_1 - 1$, and the rest of the proof is exactly same as above. Thus we leave it to the reader. This completes the proof. \square

Finally, let $\beta^{(2)} = \sum_i \left(\sum_{j=0}^2 c_j^i \xi^j \right) Y^i$ for $Y = 1 - t\xi^2$. By the identity (4.1) with $q = 2$ again, we obtain

$$(4.4) \quad \begin{aligned} & (1+t)^{k_0-1} (1+t\xi)^{k_1} \left(\sum_i \left(\sum_{j=0}^2 c_j^i \xi^{2j} \right) (1-t^2\xi)^i \right) \\ &= 2^l \left(\sum_i \left(\sum_{j=0}^2 c_j^i \xi^j \right) (1-t\xi^2)^i \right) (1+t^2\xi)^{n_2}. \end{aligned}$$

If we compare the coefficients of Y^i , we obtain

$$(1 + \xi)^{k_0-1} (1 + \xi^2)^{k_1} 2^i \sum_{j=0}^2 c_j^i \xi^{2j} = 2^l 2^{n_2} \sum_{j=0}^2 c_j^i \xi^j.$$

Since $k_0 + k_1 + i - 1 < l + n_2 = l + n - n_0 - n_1 = (m - 1) - n_0 - n_1 = m_2 - 1 + k_0 + k_1$, for $i \leq m_2 - 1$ we have $\sum_{j=0}^2 c_j^i = 0$. On the other hand, a similar argument as in Lemma 4.4 shows that

$$\sum_{j=0}^2 c_j^i \nu^{kj} = 0$$

for $i < l + n_2 = m_2 - 1 + k_0 + k_1$ and some k . Since $k_0 + k_1 \geq 0$ by Lemma 3.8, we have proved that $c_0^i = c_1^i = c_2^i = 0$ for all $i \leq m_2 - 2$.

Finally, if we substitute $Y = 1 - t\xi^2$ to the equation (4.4), we get

$$(4.5) \quad \begin{aligned} & (1 + \xi - Y\xi)^{k_0-1} (1 + \xi^2 - Y\xi^2)^{k_1} (c_0^{m_2-1} + c_1^{m_2-1}\xi^2 + c_2^{m_2-1}\xi) (2Y - Y^2)^{m_2-1} \\ &= 2^l (c_0^{m_2-1} + c_1^{m_2-1}\xi + c_2^{m_2-1}\xi^2) Y^{m_2-1} (2 - Y)^{n_2}. \end{aligned}$$

Now, comparing the coefficients of Y^{m_2-1} in the equation (4.5), we get

$$(4.6) \quad \begin{aligned} & 2^{m_2-1} (c_0^{m_2-1} + c_1^{m_2-1}\xi^2 + c_2^{m_2-1}\xi) (1 + \xi)^{k_0-1} (1 + \xi^2)^{k_1} \\ &= 2^{l+n_2} (c_0^{m_2-1} + c_1^{m_2-1}\xi + c_2^{m_2-1}\xi^2). \end{aligned}$$

Since $l + n_2 - m_2 + 1 = k_0 + k_1$, it follows from (4.6) and the relation $(1 + \nu)(1 + \nu^2) = 1$ that we have

$$\prod_{k=1}^2 \left(\sum_{j=0}^2 c_j^{m_2-1} \nu^{2kj} \right) = 2^{2(2-k_1)} \prod_{k=1}^2 \left(\sum_{j=0}^2 c_j^{m_2-1} \nu^{kj} \right).$$

Thus if k_1 is not equal to 2, then we have $\sum_{j=0}^2 c_j^{m_2} \nu^{kj} = 0$ for some $k = 1$ or 2 . Since ν is a generic generator of the cyclic group \mathbf{Z}_3 , this implies $c_0^{m_2-1} = c_1^{m_2-1} = c_2^{m_2-1}$. Thus β should be of the form

$$(4.7) \quad \beta = a(1 + \xi + \xi^2)(1 - t)^{m_0}(1 - t\xi)^{m_1}(1 - t\xi^2)^{m_2-1}, \quad a \in \mathbf{Z},$$

as asserted. On the other hand, if k_1 is equal to 2 then so is k_2 , and thus the spin number $\text{Spin}(e^{2\pi i/3}\tau, X)$ is non-negative. But this implies that we should have $k_0 = k_2 = 0$ by a similar argument as in Lemma 3.6, which is a contradiction. This completes the proof of Proposition 4.1. \square

Now we are ready to prove our main theorem of this section.

Theorem 4.6. *Let X be a homotopy K3 surface, and let $\tau : X \rightarrow X$ be a periodic diffeomorphism τ of odd prime order p satisfying $b_+(X/\tau) = 3$. Assume that the spin number $\text{Spin}(\hat{\tau}, X)$ is both rational and negative. Then the Seiberg-Witten invariant for the trivial spin^c structure vanishes identically.*

Proof: Since the difference of two spin structures is naturally an element of $H^1(X, \mathbf{Z}_2) = 0$, τ clearly generates a spin action of odd prime order p . Moreover, since in our case we may assume that $k_0 \leq 0$ and $k_1 = k_2 = \dots = k_{p-1}$ by Lemma 3.8, all the conditions in Proposition 4.1 are satisfied. Thus we can conclude from Proposition 4.1 that β_f should be of the form

$$\beta_f = a(1 + \xi + \dots + \xi^{p-1})(1 - t)^{m_0}(1 - t\xi)^{m_1} \dots (1 - t\xi^{p-1})^{m_{p-1}-1}, \quad a \in \mathbf{Z}.$$

Now we claim that β_f vanishes identically. To show this, for our convenience we let

$$\beta_f = \gamma(1 - t)^{m_0}(1 - t\xi)^{m_1} \dots (1 - t\xi^{p-1})^{m_{p-1}-1}, \quad a \in \mathbf{Z},$$

where $\gamma = a(1 + \xi + \dots + \xi^{p-1})$. By putting β_f into the identity (4.1), it is straightforward to see

$$(4.8) \quad \psi^q(\gamma)(1 + t + \dots + t^{q-1})^{k_0} \dots (1 + t\xi^{p-1} + \dots + t^q \xi^{q(p-1)})^{k_{p-1}-1} = q^l \gamma.$$

By plugging $\xi = 1$ and $q = p$ (say) into (4.8) and using the identity $t = 1 - T$ we should have

$$ap(1 + t + \dots + t^{p-1}) = ap\left(p - \frac{(p-1)p}{2}T + \dots\right) = p^2 a.$$

(It will not lose any generality that m is sufficiently large, so that $m > 1$ in case that the virtual dimension d of the Seiberg-Witten moduli space is zero. e.g., see Section 4 of [10].) Therefore, we see that a should be zero. This completes the proof. \square

Now we are ready to prove one of the main results.

Theorem 4.7. *Let X be a homotopy $K3$ surface, and let $\tau : X \rightarrow X$ be a periodic diffeomorphism of odd prime order p . Assume that the spin number $\text{Spin}(\hat{\tau}, X)$ is rational and negative. Then τ cannot act trivially on the self-dual part $H_+^2(X; \mathbf{R})$ of the second cohomology group.*

Proof: Suppose that τ acts trivially on the self-dual part $H_+^2(X; \mathbf{R})$ of the second cohomology group. Then we have $b_+(X/\mathbf{Z}_p) = 3$. Since the spin number $\text{Spin}(\hat{\tau}, X)$ is rational and negative by the assumption, we can apply Theorem 4.6 to derive a contradiction to the theorem of Morgan and Szabó in [21]. This completes the proof. \square

5. APPLICATIONS

The aim of this section is to give just a few applications of our main results and finally prove Theorem 1.2. To do so, recall some necessary facts about pseudofree actions. An action is called *pseudofree* if it is free on the complement of a discrete subset. Now assume at the moment that the action of \mathbf{Z}_3 is pseudofree. When we fix a generator of a cyclic group \mathbf{Z}_3 , the representation at each isolated fixed point can be determined by a pair of non-zero integers (α, β) modulus 3 which is well-defined up to order and signs. Thus there are only two types of (α, β) : $(1, 2)$ and $(1, 1)$. Let f_1 (resp. f_2) be the number of fixed points of type $(1, 2)$ (resp. $(1, 1)$). Then by the Atiyah-Singer G -signature theorem it is easy to find

$$(5.1) \quad 3\sigma(X/\mathbf{Z}_3) = \sigma(X) + \frac{2}{3}(f_1 - f_2).$$

Since we have

$$(5.2) \quad 3\chi(X/\mathbf{Z}_3) - 2(f_1 + f_2) = \chi(X),$$

it follows from (5.1) and (5.2) that we have

$$4f_1 + 2f_2 = 9b_+(X/\mathbf{Z}_3) - 3.$$

But $b_+(X/\mathbf{Z}_3)$ is either 1 or 3. Hence we have $2f_1 + f_2 = 3$ or $2f_1 + f_2 = 12$. On the other hand, notice that since the signature $\sigma(X/\mathbf{Z}_3)$ is always integer, $f_1 - f_2 \equiv 6 \pmod{9}$. This fact will not be used in this paper, but will be useful when we try to classify cyclic group actions of order 3 on $K3$ surfaces.

Theorem 5.1. *Let X be a homotopy $K3$ surface, and let $\tau : X \rightarrow X$ be a periodic diffeomorphism of order 3. Assume that the fixed point set is isolated. Then τ cannot act trivially on cohomology.*

Proof: Suppose that τ acts trivially on cohomology. Then it follows from (5.1) that $-16 = \sigma(X) = \frac{1}{3}(f_1 - f_2)$. But the spin number $\text{Spin}(\hat{\tau}, X)$ is the same as $\frac{1}{3}(f_1 - f_2)$ by the Atiyah-Singer G -spin theorem. Thus this number is always negative. Now if we combine the Theorem 4.7 together with Corollary 3.3, we are done. This completes the proof. \square

Moreover, we can show that a homotopy $K3$ surface admits a periodic diffeomorphism of order 3 which acts trivially on cohomology only if its spin number is negative. Hence we can give the following

Theorem 5.2. *Let X be a homotopy $K3$ surface, and let $\tau : X \rightarrow X$ be a periodic diffeomorphism of order 3. Then τ cannot act trivially on cohomology.*

Proof: By the Atiyah-Singer G -signature theorem (e.g., see Proposition 6.18 of [3]) we have

$$(5.3) \quad 3\sigma(X/\mathbf{Z}_3) = \sigma(X) + \sum_k \sum_{l=1}^2 \csc^2(\pi l/3) \langle [F_k], [F_k] \rangle + \frac{2}{3}(f_1 - f_2).$$

Moreover, since the spin number $\text{Spin}(\hat{\tau}, X)$ is given by

$$(5.4) \quad \frac{1}{4} \sum_k \cos(\pi/3) \csc^2(\pi/3) \langle [F_k], [F_k] \rangle + \frac{1}{3}(f_1 - f_2)$$

by the Atiyah-Singer G -spin theorem, we have

$$(5.5) \quad \begin{aligned} 3\text{Ind}_{\mathbf{Z}_3} D &= -\frac{\sigma(X)}{8} + 2\text{Spin}(\hat{\tau}, X) \\ &= -\frac{\sigma(X)}{8} + \frac{1}{4} \sum_k \sum_{l=1}^2 (-1)^l \cos(\pi l/3) \csc^2(\pi l/3) \langle [F_k], [F_k] \rangle \\ &\quad + \frac{2}{3}(f_1 - f_2). \end{aligned}$$

Here the signs in the spin number are determined as in the paper [2] of Atiyah and Hirzebruch.

Now assume that τ acts trivially on cohomology. Then it follows from (5.3) and (5.5) that we have

$$(5.6) \quad k_0 = 2 + \frac{1}{4}(f_1 - f_2),$$

where we use the identity $k_0 = \text{Ind}_{\mathbf{Z}_3} D$ and

$$\sum_{l=1}^{p-1} (-1)^l \cos(\pi l/p) \csc^2(\pi l/p) = -\frac{1}{2} \sum_{l=1}^{p-1} \csc^2(\pi l/p).$$

If $f_1 = f_2$ then it follows from (5.4) that the spin number $\text{Spin}(\hat{\tau}, X)$ becomes

$$\frac{1}{6} \sum_k \langle [F_k], [F_k] \rangle.$$

It is known in [9] and [20] that the fixed point set of each group element consists of 2-spheres and/or isolated fixed points. Hence by the adjunction inequality, the self-intersection number $\langle [F_k], [F_k] \rangle$ is always less than or equal to 0. Thus the spin number $\text{Spin}(\hat{\tau}, X)$ is non-positive in this case. But if $f_1 = f_2$, then it also follows from (5.6) that $k_0 = 2$, which implies that the spin number is positive. Hence we can conclude that this case does not happen. Notice that $\text{Spin}(\hat{\tau}, X)$ cannot be zero (e.g., see (3.6)).

Next if $f_1 \neq f_2$, then using $k_0 \leq 2$ by Lemma 3.4 we have $k_0 < 2$, which implies that the spin number $\text{Spin}(\hat{\tau}, X)$ is negative. Hence in this case we can apply Theorem 4.6 to obtain the vanishing of the Seiberg-Witten invariant for the trivial spin^c structure. This is a contradiction, which completes the proof. \square

In fact, there exists an example of a $K3$ surface admitting a homologically nontrivial action of \mathbf{Z}_3 whose spin number is positive.

Example 5.3. Consider the Fermat quartic surface X which is defined by the equation $\sum_1^4 z_i^4 = 0$ in \mathbf{CP}^3 . Then define an action of \mathbf{Z}_3 on X generated by

$$[z_1, z_2, z_3, z_4] \mapsto [z_1, z_3, z_4, z_2].$$

Then the \mathbf{Z}_3 -action has 6 isolated fixed points: four points of the form $[1, a, a, a]$ with $1 + 3a^4 = 0$ and two more points of the form $[0, 1, b^2, b]$ with $b^2 + b + 1 = 0$. In our case $f_1 + f_2 = 6$. Thus using the identity $2f_1 + f_2 = 12$, we have $f_1 = 6$ and $f_2 = 0$. On the other hand, the spin number $\text{spin}(\hat{\tau}, X)$ is given by $\frac{1}{3}(f_1 - f_2)$, if the fixed point set is isolated. Thus in our case the spin number $\text{Spin}(\hat{\tau}, X)$ is 2 which is positive. It is also easy to see that $b_+(X/\mathbf{Z}_3) = 3$ and $b_-(X/\mathbf{Z}_3) = 7$. So the group action is nontrivial on $H^2(X, \mathbf{R})$.

REFERENCES

- [1] M. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes: II. Applications*, Ann. Math. **88** (1968), 451–491.
- [2] M. Atiyah and F. Hirzebruch, *Spin manifolds and group actions*, Essays in Topology and Related Topics, Springer-Verlag (1970), 18–28.
- [3] M. Atiyah and I. Singer, *The index of elliptic operators: III*, Ann. Math **87** (1968), 546–604.
- [4] J. Bryan, *Seiberg-Witten theory and $\mathbf{Z}/2^p$ actions on spin 4-manifolds*, Math. Res. Lett. **5** (1998), 165–183.
- [5] W. Chen, *Pseudoholomorphic curves in four-orbifolds and some applications*, preprint (2004); ArXiv:math.SG/0410608.
- [6] W. Chen and S. Kwasik, *Symplectic symmetries of 4-manifolds*, preprint (2005).
- [7] S. Donaldson, *The Seiberg-Witten equations and 4-manifold topology*, Bull. Amer. math. Soc. **33** (1996), 45–70.
- [8] A. Edmonds, *Construction of group actions on four manifolds*, Trans. Amer. Math. Soc. **299** (1987), 155–170.
- [9] A. Edmonds, *Aspects of group actions on four-manifolds*, Topology and its Appl. **31** (1989), 109–124.
- [10] F. Fang, *Smooth group actions on 4-manifolds and Seiberg-Witten invariants: II*, Int. J. Math. **9** (1998), 957–973.
- [11] M. Furuta, *Monopole equation and the $\frac{11}{8}$ -conjecture*, Math. Res. Lett. **8** (2001), 279–291.
- [12] ———, *Stable homotopy version of Seiberg-Witten invariant*, MPI preprint (1997).
- [13] F. Hirzebruch, *The signature theorem: Reminiscences and Recreation*, Ann. Math. Stud. **70**, Princeton Uni. Press (1971).
- [14] J.H. Kim, *On spin $\mathbf{Z}/2^p$ -actions on spin 4-manifolds*, Topology and its Appl. **108** (2000), 197–215.
- [15] R. Kirby, *Problems in low-dimensional topology*, Berkeley, 1985.
- [16] S. Kobayashi, *Transformation groups in differential geometry*, Springer, 1995.
- [17] D. Kotschick, *The Seiberg-Witten invariants of symplectic four-manifolds (after C.H. Taubes)*, Séminaire Bourbaki **48**ème année (1995-6), 195–220.
- [18] B. Lawson and M. Michelsohn, *Spin geometry*, Princeton University Press, 1989.
- [19] T. Matumoto, *Homologically trivial smooth involutions on K3-surfaces*, Aspects of low-dimensional manifolds, Adv. Studies in Pure Math. **20** (1992), 365–374.
- [20] M. McCooey, *Symmetry groups of four-manifolds*, preprint (1999): arXiv:math.GT/9907180.
- [21] J. Morgan and Z. Szabó, *Homotopy K3 surfaces and Mod 2 Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), 17–21.
- [22] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183–221.
- [23] V. Nikulin, *Finite groups of automorphisms of Kählerian surfaces of type K3*, Moscow Math. Soc. **38** (1980), 71–137.
- [24] C.A.M. Peters, *Holomorphic automorphisms of compact Kähler surfaces and their induced actions on cohomology*, Inven. Math. **52** (1979), 143–148.
- [25] D. Ruberman, *Involutions on spin 4-manifolds*, Proc. Amer. Math. Soc. **123** (1995), 593–596.
- [26] C. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994), 809–822.

DEPARTMENT OF MATHEMATICS, KAIST, KUSONG-DONG, YUSONG-GU, DAEJON 305-701, SOUTH KOREA

E-mail address: jinkim11@kaist.ac.kr